Dynamic Continuum Modeling of Beamlike Space Structures Using Finite-Element Matrices

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A rational and straightforward method is introduced for developing equivalent continuum models of large beamlike periodic lattice structures based on energy equivalence. The extended Timoshenko beam model is chosen to account for coupling among extension, transverse shear, and bending deformations. The procedure for developing continuum models involves using existing finite-element matrices in calculating the strain and kinetic energies of a repeating cell. These results are then used to calculated the reduced equivalent continuum stiffness and mass matrices. The reduced finite-element stiffness and mass matrices are derived to provide a direct comparison with the reduced equivalent continuum stiffness and mass matrices. The equivalent continuum beam properties are obtained from the direct comparison. The numerical results of free vibration analysis show that the method developed in this paper gives very reliable dynamic characteristics compared to other methods.

Nomenclature

$[B]$ = matrix in Eq. (19)
$\{b\}$ = constants to be determined by boundary conditions
C_i = elastic coupling coefficients ($i = 1, 2, 3$)
$\{D\}$ = continuum displacement vector
$\{d\}$ = lattice displacement vector
EA = longitudinal rigidity
EI = bending rigidity
GA = transverse shear rigidity
$[H]$ = matrix defined in Table 1
h = beam thickness
I = mass moment of inertia of the beam cross section
$[I]$ = unit matrix
$[K]$ = equivalent continuum stiffness matrix
$[\bar{K}]$ = reduced equivalent continuum stiffness matrix
$[k]$ = element stiffness matrix
$[L]$ = locator matrix
M = bending moment
$[M]$ = equivalent continuum mass matrix
$[\bar{M}]$ = reduced equivalent continuum mass matrix
m = mass density per unit length
$[m]$ = element mass matrix
N = number of continuum degrees of freedom, extensional force
Q = transverse shear force
R = dynamic inertia property defined in Appendix
$[R]$ = nodal continuum transformation matrix
$[S]$ = coordinate transformation matrix
T = kinetic energy
u, U = longitudinal displacement
V = strain energy
w, W = transverse shear displacement
α = mode shape parameter
$\{\delta\}$ = nodal degrees of freedom of lattice model
$\{\Delta\}$ = nodal degrees of freedom of continuum model
θ, Θ = rotation of beam cross section
ω = circular frequency
$[\Omega]$ = element continuum transformation matrix
ρ = mass density per unit volume

Subscripts

C = properties of equivalent continuum beam model
e = properties of element member
ETB = properties of extended Timoshenko beam model
N = natural frequency
left = properties at left beam cross section
right = properties at right beam cross section

Received Jan. 30, 1989; revision received June 8, 1989.

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Introduction

LARGE lattice structures are prime candidates for future space applications due to their low cost, light weight, high stiffness, as well as ease of packaging, transporting and assembling in space. Large space structures such as space stations, space mirrors, and deployable antenna systems would be assembled to form lattice-type structures having dimensions on the order of $10^2 \sim 10^3$ m. Detailed aspects of large space structures have been discussed in recent issues of *Astronautics & Aeronautics*.[1,2] The American Society of Civil Engineers has published extensive historical reviews of the research activities on space structures prior to 1975.[3-5]

Structural and dynamic characteristics of large lattice structures must be predicted accurately during the initial design phase since they cannot be tested full scale in their operational environments prior to flight. Conventional finite-element analysis for large lattice structures may be too expensive and time consuming in order to obtain reliable solutions because of structural flexibility and large size. Thus, special techniques to cope with the very large number of elements and nodes within a structure are essential to obtain successful finite-element solutions. This is especially true for high-frequency modes. Alternative methods[6-25] have been developed for simplified structural modeling of lattice structures instead of the finite-element method. Of these methods, the approximation of a periodic lattice structure (simply lattice model) by the equivalent continuum model (simply continuum model) is known to provide a very promising and practical solution method for overall vibration modes and structural response.

The key to continuum modeling involves the determination of appropriate relationships between the geometric and material properties of the lattice and continuum models. Hence, the continuum method is not unique and can fall in one of several distinct categories. They are 1) relating force or deformation characteristics of a repeating cell of lattice model to those of the continuum model,[10,11] 2) converting finite difference equations from the discrete field method into approximate differential equations through the use of Taylor series,[12] 3) using energy equivalence between lattice and continuum models,[13–18] 4) averaging the contribution of each unidirectional structural element to the overall structural stiffness,[19,20] and 5) conducting static or experimental analysis for a repeating cell of the lattice model to measure its structural properties[21–25] (conceptually similar to 1). In point 3 "equivalence" means that the lattice and continuum models contain equal kinetic and strain energies when both are subject to the same displacement and velocity fields along the centerlines.

The last three approaches have shown to give good results when the wavelength of a vibration mode spans many repeating cells. However, they have some shortcomings from the viewpoint of practical application. In the method of Nayfeh and Hefzy,[20] a reduction procedure must be followed to obtain the stiffness of a one- or two-dimensional structure from a three-dimensional one. The methods of Noor et al.,[14] and Dow et al.[16] methods deal with complicated kinematic quantities such as strain gradients in calculating strain energy and in matching geometric constraints. Finally, the approach Sun et al.[21] and Sun and Kim[22] requires proper force placement on a repeating cell prior to static analysis. Thus it seems that a simple and accurate modeling method not requiring complicated procedures would certainly be desirable and in some respects mandatory.

In the present paper, a rational and straightforward method is developed for the beamlike periodic lattice structures based on energy equivalence. In order to evaluate this performance of the method developed, free vibration analyses are conducted for several types of lattice models.

Development of Dynamic Continuum Models

It is well known that shear deformation is frequently important when beam theory is used to model systems such as trusses or lattice structures.[26] Thus the Timoshenko beam (TB) model represents a powerful tool for the modeling and analysis of large lattice structures. Conventional Timoshenko beam theory was developed for homogeneous isotropic materials in which extension, transverse shear, and bending deformations are uncoupled. When a beamlike structure is not symmetric with respect to its midplane, coupling among these three basic deformations may be significant. Therefore, this paper deals with the extended Timoshenko beam (ETB) model[22] instead of the classical model.

The accuracy and ease of application of energy equivalence techniques generally depends on the way in which the strain and kinetic energies are calculated. It is well known that finite-element stiffness and mass matrices can be derived from the strain and kinetic energies calculated for a finite-element after making a reasonable hypothesis for the displacement fields. Therefore, it seems reasonable to use these finite-element matrices in calculating the strain and kinetic energies of a repeating lattice cell. The finite-element stiffness and mass matrices for axial-bar and beam elements are available from many text books. In the method developed herein, we will use them directly for the calculation of strain and kinetic energies stored in a repeating cell. Briefly, the modeling procedure consists of the following steps.

1) Isolate a repeating cell from the lattice model.

2) Introduce a continuum transformation matrix which expresses the nodal displacement vector of a lattice element (simply lattice displacement vector) in terms of the nodal displacement vector of the continuum model (simply continuum displacement vector).

3) Derive element strain and kinetic energies stored in each lattice element in terms of the continuum displacement vector. Well-defined existing finite-element stiffness and mass matrices are used.

4) Derive total energies stored in a repeating cell by summing all element energies, which are in turn expressed as functions of the continuum displacement vector. From these energy expressions, equivalent continuum stiffness and mass matrices are derived.

5) Using the reduction methods defined in this paper, derive the reduced stiffness and mass matrices from the equivalent continuum matrices of the lattice model and also from the finite-element matrices of the continuum model. The finite-element matrices for the present model are derived in the Appendix.

6) Equate the reduced stiffness and mass matrices of the lattice model with those of the continuum model to determine the equivalent continuum structural properties.

In the present paper, we consider lattice structures with fixed cross sections. The lattice structures can be composed of several different types of lattice elements: longitudinal, diagonal, and batten bars. The geometry of a typical repeating cell isolated from a lattice model is shown in Fig. 1. Figure 2 shows an equivalent continuum beam model to be transformed from the repeating cell following the preceding modeling procedure.

Assuming small amplitute vibrations, the degrees of freedom (DOF) at each node of a repeating cell (called lattice DOF), $\{\delta\}$, can be expressed as a function of the nodal DOF of the continuum model (called continuum DOF), $\{\Delta\}$, by introducing the nodal-continuum transformation matrix $[R]$ defined by

$$\{\delta\} = [R]\{\Delta\} \tag{1}$$

Table 1 shows the details of Eq. (1) for the three-dimensional deformation. When the lattice model displays only plane motion, the out-of-plane DOF such as v, p, r, V, P, R and related terms in $[R]$ will vanish from Table 1.

Considering a lattice element with ith and jth nodes, we can construct lattice displacement vectors as follows:

$$\{d\} = \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \begin{Bmatrix} [R_i] \; \{\Delta_i\} \\ [R_j] \; \{\Delta_j\} \end{Bmatrix} \tag{2}$$

The continuum DOF at both ends of the continuum model are also combined to form a continuum displacement vector in the form

$$\{D\} = \begin{Bmatrix} \{\Delta_{\text{left}}\} \\ \{\Delta_{\text{right}}\} \end{Bmatrix} \tag{3}$$

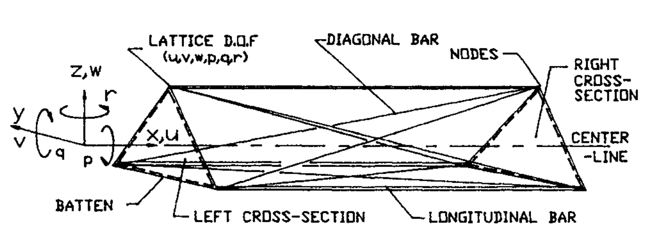

Fig. 1 Geometry of a typical repeating cell: single-bay, double-laced lattice structure.

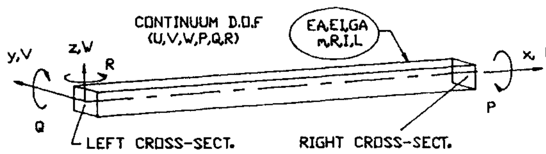

Fig. 2 Equivalent continuum beam model.

Table 1 Transformation to continuum degrees of freedom

Transformation rule			$\{\delta\} = [R]\{\Delta\}$
Lattice element	Latice DOF$\{\delta\}$	Continuum DOF$\{\Delta\}$	Nodal continuum transform. matrix $[R]$
Axial bar element	$\{u,v,w\}$	$\{U,V,W, P,Q,R\}$	$\left[I \;\middle|\; \begin{matrix} 0 & z & -y \\ -z & 0 & 0 \\ y & 0 & 0 \end{matrix} \right] = [I \mid H]$
Beam element	$\{u,v,w, p,q,r\}$	$\{U,V,W, P,Q,R\}$	$\begin{bmatrix} I & H \\ 0 & I \end{bmatrix}$

By introducing a locator matrix $[L]$, which identifies the location of the cross section in which a node is located, the continuum DOF $\{\Delta\}$ can be represented in terms of the continuum displacement vector as

$$\{\Delta\} = [L]\{D\} \tag{4}$$

Combining Eqs. (3) and (4), the lattice displacement vector for each lattice element can be represented in terms of the continuum displacement vector as

$$\{d\} = [\Omega]\{D\} \tag{5}$$

where $[\Omega]$ is defined by

$$[\Omega] = \begin{bmatrix} [R_i][L_i] \\ [R_j][L_j] \end{bmatrix} \tag{6}$$

The finite-element stiffness matrix $[\bar{k}]$ and mass matrix $[\bar{m}]$ for a lattice element are obtained from

$$[\bar{k}] = [S]^T[k][S], \qquad [\bar{m}] = [S]^T[m][S] \tag{7}$$

where $[k]$ and $[m]$ are the finite-element stiffness and mass matrices referred to an element reference frame, and $[S]$ is the transformation matrix which transforms the element coordinates to the global coordinates.

The element strain and kinetic energies of a lattice element are given by

$$V_e = \tfrac{1}{2}\{d\}^T[\bar{k}]\{d\}$$

$$T_e = \tfrac{1}{2}\{\dot{d}\}^T[\bar{m}]\{\dot{d}\} \tag{8}$$

By substituting Eqs. (5) and (7) into Eqs. (8), the element energies are expressed in terms of the continuum displacement vector as follows:

$$V_e = \tfrac{1}{2}\{D\}^T[K_e]\{D\}$$

$$T_e = \tfrac{1}{2}\{\dot{D}\}^T[M_e]\{\dot{D}\} \tag{9a}$$

where

$$[K_e] = [\Omega]^T[\bar{k}][\Omega], \qquad [M_e] = [\Omega]^T[\bar{m}][\Omega] \tag{9b}$$

Summing all element energies, the total energy stored in a repeating cell is calculated from

$$V = \Sigma V_e = \tfrac{1}{2}\{D\}^T[K]\{D\}$$

$$T = \Sigma T_e = \tfrac{1}{2}\{\dot{D}\}^T[M]\{\dot{D}\} \tag{10a}$$

where

$$[K] = \Sigma[K_e], \qquad [M] = \Sigma[M_e] \tag{10b}$$

Since Eq. (10a) is represented as the functions of the continuum displacement vector, Eq. (10b) is identified as the equivalent continuum stiffness and mass matrices of the lattice model. They are $2N$ by $2N$ symmetric matrices when the continuum DOF is N. Direct comparison of these matrices with the finite-element matrices of the ETB model (Table 5) may give $N(1 + 2N)$ relations, which is usually larger than the number of equivalent continuum beam properties to be determined. Thus, a simple and rational scheme is required to reduce the excessive number of relations as far as possible. The reduction procedure may not be unique in its broadest sense. In this paper, however, two reduction methods are introduced so that the reduction method dependence of numerical results can be investigated. This gives valuable insight into the optimal design of a reduction method that yields better numerical results.

First, the stiffness and mass matrices for both the lattice model [Eq. (10b)] and the ETB model (Table 5) are partitioned into the following forms:

$$[K] = \begin{bmatrix} K_1 & K_2 \\ K_2^T & K_3 \end{bmatrix}, \qquad [M] = \begin{bmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{bmatrix} \tag{11}$$

where K_1, K_2, K_3 and M_1, M_2, M_3 are submatrices of equal dimension. In these expressions $K_2^T$and M_2^T represent the transposes of K_2 and M_2. Two reduction methods are proposed to generate reduced stiffness and mass matrices. These are:

Method 1:

$$[\bar{K}] = \tfrac{1}{4}[K_1 + K_3 - K_2 - K_2^T] \tag{12a}$$

$$[\bar{M}] = \tfrac{1}{3}[M_1 + M_3 + M_2 + M_2^T] \tag{12b}$$

Method 2

$$[\bar{K}] = \tfrac{1}{2}[K_1 + K_3] \tag{13a}$$

$$[\bar{M}] = \tfrac{1}{2}[M_1 + M_3] \tag{13b}$$

The factors ¼, ⅓, and ½ are adopted in Eqs. (12) and (13) so that these "averaged" matrices approximately resemble the diagonal submatrices of Eq. (11). The reduced matrices are also symmetric. Thus, they yield $N(1 + N)/2$ relations reduced by the order of $N(1 + N)/2(1 + 2N)$. The extended Timoshenko beam model, which takes plane motion with three continuum DOF ($N = 3$), as an instance, has six structural properties: longitudinal rigidity (EA), transverse shear rigidity (GA), bending rigidity (EI), together with three elastic coupling terms (C_i).[22] Since there are six relations for three continuum DOF, the six structural properties can be determined by forcing $[\bar{K}]_C = [\bar{K}]_{ETB}$. Similary, three dynamic inertia properties are obtained from the relation $[\bar{M}]_C = [\bar{M}]_{ETB}$.

Free Vibration Analysis

Once the equivalent continuum beam properties for a lattice model are found, the free vibration analysis of the equivalent continuum model is quite straight-forward. In this paper, we confine our discussion to planar motion. The free vibration equations of motion for the ETB model are as follows:

$$\frac{d}{dx}\begin{Bmatrix} N \\ Q \\ M \end{Bmatrix} = \begin{bmatrix} m & 0 & R \\ 0 & m & 0 \\ R & 0 & I \end{bmatrix}\begin{Bmatrix} \ddot{u} \\ \ddot{w} \\ \ddot{\theta} \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ Q \end{Bmatrix} \tag{14}$$

The force-deformation relations that account for the elastic couplings between deformations are given by

$$\begin{Bmatrix} N \\ Q \\ M \end{Bmatrix} = \begin{bmatrix} EA & C_1 & C_2 \\ C_1 & GA & C_3 \\ C_2 & C_2 & EI \end{bmatrix}\begin{Bmatrix} u_{,x} \\ w_{,x} + \theta \\ \theta_{,x} \end{Bmatrix} \tag{15}$$

Based on the approach in Ref. 22, we assume harmonic solutions in the form of

$$\begin{Bmatrix} u \\ w \\ \theta \end{Bmatrix} = \begin{Bmatrix} U \\ W \\ \Theta \end{Bmatrix} e^{\alpha x} e^{i\omega t} \tag{16}$$

Substitution of Eqs. (15) and (16) into Eq. (14) yields an eigenvalue problem from which a cubic equation for α^2 is obtained. Six characteristic values of α and the corresponding eigenvectors $\{U, W, \Theta\}$ can be calculated at any frequency. The general solution is then given by

$$\begin{Bmatrix} u \\ w \\ \theta \end{Bmatrix} = \sum b_k \begin{Bmatrix} U \\ W \\ \Theta \end{Bmatrix}_k e^{\alpha_k x} e^{i\omega t} \tag{17}$$

where b_k are arbitrary constants to be determined from boundary conditions. The most frequent boundary conditions for the ETB model are

Free end (F):

$$\{N, Q, M\} = \{0\} \tag{18a}$$

Clamped end (C):

$$\{c, w, \theta\} = \{0\} \tag{18b}$$

Simply-supported end (S):

$$\{u, w, M\} = \{0\} \tag{18c}$$

Applying the appropriate boundary conditions in Eq. (17) yields simultaneous homogeneous algebraic equations of the form

$$[B(\omega)]\{b\} = \{0\} \tag{19}$$

The frequency equation is obtained from the condition that the determinent of matrix $[B]$ must vanish for the existence of nontrivial solution, $\{b\}$, i.e.,

$$|B(\omega_N)| = 0 \tag{20}$$

Iterative calculation is necessary to find the natural frequencies ω_N at which the frequency Eq. (20) is satisfied.

Evaluative Examples

A single-bay, double-laced lattice beam with 10 repeating cells (Fig. 1) and four plane trusses with 20 repeating cells (Fig. 3) are taken for the numerical investigation. Every lattice member in a repeating cell is modeled as an axial-bar element for the numerical calculation. Details of the material properties and geometric dimensions of the lattice elements used are given in Table 2. We consider two types of lattice beam. They are cantilevered (C–F) and simply supported (S–S) beams. To evaluate the accuracy and validity of the present continuum method, equivalent continuum beam properties, natural frequencies, and mode shapes are compared to those by other methods: the conventional full-scale finite-element analysis and of Refs. 14, 21, and 22. continuum methods.

Table 3 compares equivalent continuum beam properties for the plane trusses; results from the two present methods are compared with those of Refs. 21 and 22. Although it is not possible to judge which method is better by direct comparison of equivalent continuum beam properties only, it may be noted that the present method and the methods of Refs. 21 and 22 give quite similar results with the elastic coupling terms for the unsymmetric plane trusses. The accuracy of the natural

Table 2 Material properties and geometric dimensions of lattice members used in the present study

Lattice members	ρ (kg/m^3)	E (N/m^2)	L (m)	A (m^2)	Designation
Long bar	2768	71.7×10^9	7.5	8×10^{-5}	
				18×10^{-5}	
Diagonal bar	2768	71.7×10^9	9.0	4×10^{-5}	
Batten	2768	71.7×10^9	5.0	6×10^{-5}	

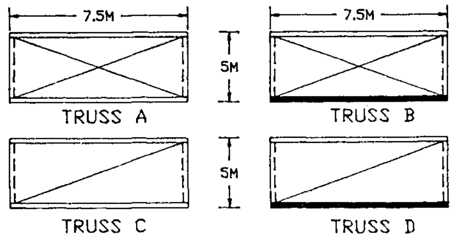

Fig. 3 Repeating cells of the plane trusses.

frequencies obtained from the present methods is shown in Fig. 4. Comparisons are made with finite-element solutions using SAP-IV when the truss is modeled using axial-bar elements. The natural frequencies obtained are very accurate; within 3% for the fundamental modes. Figure 4 indicates that the finite-element solutions are usually between those of method 1 and method 2 at lower frequency modes and they are closer to the results of method 1 for the symmetric and quasisymmetric plane trusses.

On the other hand, the present methods generally overestimate natural frequencies in higher frequency modes. This is a problem that has been recognized before and is one that has to be overcome in continuum modeling it. The tendency to overestimate natural frequencies seems to be more visible when the plane truss is unsymmetric with respect to its midplane. This is because of the nonbeamlike local effects of the lattice model that become important for the unsymmetric plane trusses at high frequencies. In general, the continuum model does not specify the detailed information within a repeating cell and therefore does not represent local effects accurately. The finite-element method gives upper bounds on natural frequencies. Hence, actual natural frequencies are expected to be slightly lower than finite-element solutions. As Berry et al.[17] showed continuum models overestimate the bending stiffness and therefore the bending frequencies increase. More reliable and easy prediction of natural frequencies, therefore, may be achieved by using method 1 for the fundamental modes and method 2 for higher frequency modes. The five least bending modes are compared in Fig. 5. On the whole, the present methods show very good agreement with the finite-element analysis.

For the single-bay, double-laced lattice beam, the equivalent continuum beam properties and natural frequencies are calculated in Table 4. The effect of centerline location in the cross section of the lattice structure is considered in two cases. The first case is when the centerline passes through the centroid of cross section (one-third height for the triangular cross section, for instance). The other case is the half-height of cross section.

For the centroid case, the present results are compared to Noor's continuum method results which has been recognized as a powerful method for formulating the equivalent continuum model of periodic lattice structures. Noor's method was applied to the same problem considered by Berry.[17] In Table 4, natural frequencies are found to be very close to one another for overall vibration modes except for the axial modes. To predict satisfactory natural frequencies for the axial modes, a

Table 3 Comparison of equivalent continuum beam properties for the plane trusses

Trusses	Truss A			Truss B			Truss C		
	Present		Sun	Present		Sun	Present		Sun
Methods	[1]	[2]	[Ref. 21]	[1]	[2]	[Ref. 21]	[1]	[2]	[Ref. 21]
EA [$\times 10^6 N$]	14.80	14.80	14.60	20.30	20.30	20.10	13.10	12.10	12.90
GA [$\times 10^6 N$]	1.47	1.47	1.47	0.73	0.73	0.66	0.73	0.73	0.66
EI [$\times 10^6 N - m^2$]	71.70	71.70	71.70	11.70	11.70	11.70	71.70	71.70	71.70
C_1 [$\times 10^6 N$]	0.00	0.00	0.00	1.10	1.10	0.99	1.10	1.10	0.99
C_2 [$\times 10^6 N - m$]	0.00	0.00	0.00	−17.90	−17.90	17.90	0.00	0.00	0.00
C_3 [$\times 10^6 N - m$]	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
m [Kg/m]	0.82	0.87	0.82	0.96	1.02	0.96	0.69	0.74	0.69
R [Kg]	0.00	0.00	0.00	−0.69	−0.69	−0.69	0.00	0.00	0.00
I [Kg − m]	3.55	3.55	3.55	5.01	4.24	5.01	3.28	2.90	3.28

Table 4 Equivalent continuum beam properties and natural frequencies (Hz) for the single-bay, double-laced lattice beam (a = axial modes)

Centerline location in cross section	Centroid			Half height	
	Present		Noor	Present	
Methods	1	2	et al.[14]	1	2
EA [$\times 10^7 N$]	2.71	2.71	2.53	2.71	2.71
GA [$\times 10^7 N$]	0.22	0.22	0.22	0.22	0.22
EI [$\times 10^7 N - m^2$]	8.20	8.20	8.01	9.61	9.61
C_1 [$\times 10^7$]N]	0.00	0.00	0.00	0.00	0.00
C_2 [$\times 10^7 N - m$]	0.00	0.00	0.00	−1.96	−1.96
C_3 [$\times 10^7 N - m$]	0.00	0.00	0.00	0.00	0.00
m [Kg/m]	1.79	1.96	1.79	1.79	1.96
R [Kg]	0.00	0.00	0.00	−1.30	−1.42
I [Kg − m]	5.12	4.37	5.12	6.06	5.40
Cantilever					
ω_1	0.66	0.63	0.65	0.66	0.63
ω_2	3.80	3.64	3.76	3.80	3.64
ω_3	9.51	9.12	9.44	9.51	9.12
ω_4 (a)	12.95	12.40	12.51	12.95	12.40
ω_5	16.43	15.76	16.33	16.43	15.76
ω_6	23.98	23.02	23.86	23.98	23.02
ω_7	31.82	30.53	31.68	31.82	30.53
ω_8 (a)	38.87	37.19	37.53	38.87	37.19
ω_9	39.75	38.14	39.61	39.75	38.14
ω_{10}	47.68	45.75	47.55	47.68	45.75
Simply Supported Beam					
ω_1	1.82	1.75	1.81	1.93	1.85
ω_2	6.68	6.40	6.62	6.66	6.39
ω_3	13.36	12.81	13.26	13.41	12.86
ω_4 (a)	20.91	20.05	20.76	20.71	19.86
ω_5	25.91	24.79	25.02	25.39	24.30
ω_6	28.81	27.63	28.68	28.80	27.61
ω_7	36.81	35.29	36.67	36.87	35.34
ω_8	44.80	42.94	44.67	44.67	42.82
ω_9	51.83	49.59	50.04	50.42	48.27
ω_{10} (a)	52.74	50.54	52.61	52.74	50.53

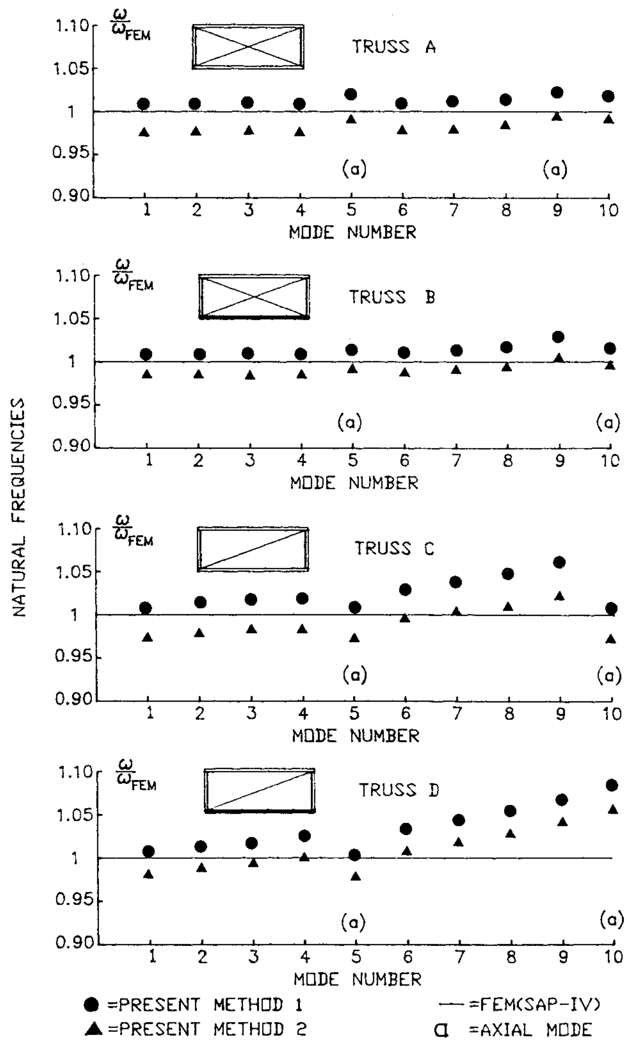

Fig. 4 Accuracy of natural frequencies for the plane trusses.

more careful investigation of the reduction methods, Eqs. (12) and (13), seems to be necessary. Figure 6 gives an indication of the accuracy of the fundamental bending modes for both cantilevered and simply supported beams. Generally, the present methods show very good agreement with Noor's method.

In case of the half height, there are not significant differences in the predictions of natural frequencies (within 5% errors for S-S beam) and mode shapes from the centroid case. Thus, a comment can be made at this point about the comparison of the equivalent continuum beam properties generated for both centroid and half-height cases. As can be seen in Table 4, the elastic coupling coefficients and dynamic inertia properties may be dependent on the selection of the beam centerline location; nevertheless, the location dependence of equivalent continuum beam properties does not influence the accurate prediction of vibration characteristics significantly.

For both centroid and half-height cases, the results from Noor's method are bounded by the results of present two methods. As in the case of plane-truss problems, Noor's results are likely to be closer to method-1 for lower frequency modes and to method-2 for higher frequency modes. Based on the numerical results, an optimal (or better) reduction method may be obtained by the proper combination of the two present reduction methods.

Many authors[14,17] have noted that continuum method becomes more accurate as the number of repeating cells increases. Physically, the number of repeating cells per wavelength increases as the total length of lattice structure increases. Thus, the wavelength of a mode spans more repeating cells so that the effect of the nonbeam-like characteristics of lattice structure become less important. Moreover, most

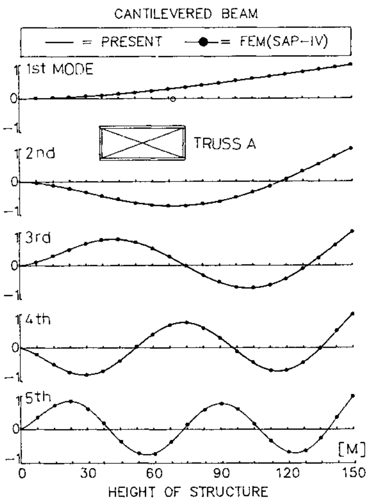

Fig. 5 Lowest five bending normal modes for the truss A.

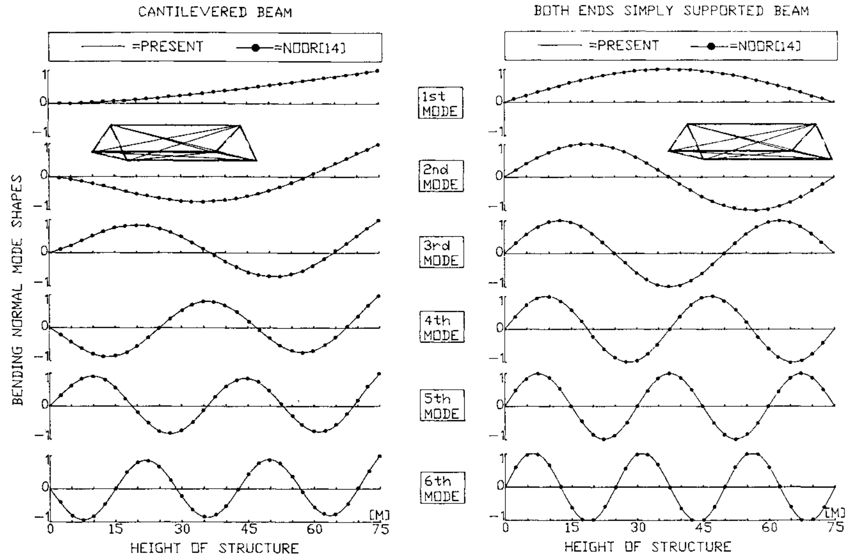

Fig. 6 Lowest six bending normal modes for the single-bay, double laced lattice beam.

large space structures are expected to operate at low frequencies. Thereby, the author believes that the method developed in this paper will provide satisfactory continuum models for large flexible beam-like lattice structures in space.

Conclusions

A simple and rational procedure for developing equivalent continuum models for periodic beamlike lattice structures has been presented. The procedure is based on the concept of energy equivalence between lattice and continuum models. The key to the procedure is the calculation of strain and kinetic energies in repeating cells in terms of a continuum displacement vector based on the finite-element matrices. Continuum transformation matrices are introduced to transform the lattice displacement vector into the continuum displacement vector.

Free vibration analyses for a single-bay, double-laced lattice beam and four plane trusses are presented. Numerical results illustrate that the present continuum method gives very accurate and competitive structural and dynamic characteristics compared to other methods. In addition, it is found that finite-element solutions and the results of Noor et al.[14] are bounded by the results of the present two methods for the fundamental modes, and the location of the centerline does not influence the natural frequencies and mode shapes of a lattice structure significantly.

Appendix: Derivation of Finite-Element Stiffness and Mass Matrices for the Extended Timoshenko Beam Model

We consider a finite element of the extended Timoshenko beam model which takes only plane motion, as shown in Fig. A1. The force-displacement relations are given by Eq. (15). The displacement fields within the finite element can be derived by solving the static equilibrium equations

$$N_{,x} = 0, \qquad Q_{,x} = 0, \qquad M_{,x} - Q = 0 \tag{A1}$$

with boundary conditions specified at the ends as

$$u(0) = U_1, \qquad w(0) = W_1, \qquad \theta(0) = \Theta_1$$

$$u(L) = U_2, \qquad w(L) = W_2, \qquad \theta(L) = \Theta_2 \tag{A2}$$

After some algebraic manipulation, the displacement fields are obtained as

$$u = (1-\beta)U_1 + \beta U_2 \tag{A3a}$$

$$\begin{aligned} w = \frac{1}{1+\Phi}\Bigg\{ & \left[(1-3\beta^2+2\beta^3) + (1-\beta)\Phi - \frac{6}{L}(\beta-\beta^2)\mu_3\right]W_1 \\ & + \left[(\beta-3\beta^2+2\beta^3)\mu_1 - \frac{6}{L}(\beta-\beta^2)\mu_2\right]U_1 + \Bigg[(-\beta+2\beta^2 \\ & -\beta^3) - \frac{1}{2}(\beta-\beta^2)\Phi + \frac{2}{L}(2\beta-3\beta^2+\beta^3)\mu_3\Bigg]L\Theta_1 \\ & + \left[(3\beta^2-2\beta^3) + \beta\Phi + \frac{6}{L}(\beta-\beta^2)\mu_3\right]W_2 \\ & + \left[(-\beta+3\beta^2-2\beta^3)\mu_1 + \frac{6}{L}(\beta-\beta^2)\mu_2\right]U_2 \\ & + \left[(\beta^2-\beta^3) + \frac{1}{2}(\beta-\beta^2)\Phi + \frac{2}{L}(\beta-\beta^3)\mu_3\right]L\Theta_2\Bigg\} \end{aligned} \tag{A3b}$$

$$\begin{aligned} \theta = \frac{1}{1+\Phi}\Bigg\{ & \frac{6}{L}(\beta-\beta^2)W_1 + \frac{6}{L}(\beta-\beta^2)\mu_1 U_1 \\ & + \left[(1-4\beta+3\beta^2) + (1-\beta)\Phi + \frac{6}{L}(\beta-\beta^2)\mu_3\right]\Theta_1 \\ & - \frac{6}{L}(\beta-\beta^2)W_2 - \frac{6}{L}(\beta-\beta^2)\mu_1 U_2 \\ & + \left[(-2\beta+3\beta^2) + \beta\Phi - \frac{6}{L}(\beta-\beta^2)\mu_3\right]\Theta_2\Bigg\} \end{aligned} \tag{A3c}$$

where $\beta = x/L$, $\Phi = \phi - 12\mu_3^2/L^2$, $\phi = 12EI/L^2GA$, $\mu_1 = C_1/GA$, $\mu_2 = C_2/GA$ and $\mu_3 = C_3/GA$. These formulas become identical to those of the conventional Timoshenko beam when we neglect coupling terms between extension, transverse shear, and bending deformations.

The strain and kinetic energies are

$$V = \frac{1}{2}\int_0^L \left[\frac{N^2}{EA} + \frac{Q^2}{GA} + \frac{M^2}{EI}\right] dx = \frac{1}{2}\{D\}^T[K]_{\rm ETB}\{D\} \tag{A4a}$$

$$T = \frac{1}{2}\left[m\int_0^L(\dot{u}^2 + \dot{w}^2)\,dx + I\int_0^L \dot{\theta}^2\,dx\right] + 2R\int_0^L \dot{u}\dot{\theta}\,dx$$

Table 5 Finite element stiffness and consistent mass matrices for the extended Timoshenko beam

Stiffness Matrix

	U_1	W_1	Θ_1	U_2	W_2	Θ_2
U_1	$\frac{EA}{L}$	$\frac{\phi C_1}{\psi L}$	$-\frac{\phi C_1}{2\psi}+\frac{C_2}{L}$	$-\frac{EA}{L}$	$-\frac{\phi C_1}{\psi L}$	$-\frac{\phi C_1}{2\psi}-\frac{C_2}{L}$
W_1		$\frac{12EI}{\psi L^3}$	$-\frac{6EI}{\psi L^2}+\frac{\phi C_3}{\psi L}$	$-\frac{\phi C_1}{\psi L}$	$-\frac{12EI}{\psi L^3}$	$-\frac{6EI}{\psi L^2}-\frac{\phi C_3}{\psi L}$
Θ_1			$\frac{(3+\psi)EI}{\psi L}-\frac{\phi C_3}{\psi}$	$\frac{\phi C_1}{2\psi}-\frac{C_2}{L}$	$\frac{6EI}{\psi L^2}-\frac{\phi C_3}{\psi L}$	$\frac{(3-\psi)EI}{\psi L}$
U_2	SYMMETRIC			$\frac{EA}{L}$	$\frac{\phi C_1}{\psi L}$	$\frac{\phi C_1}{2\psi}+\frac{C_2}{L}$
W_2					$\frac{12EI}{\psi L^3}$	$\frac{6EI}{\psi L^2}+\frac{\phi C_3}{\psi L}$
Θ_2						$\frac{(3+\psi)EI}{\psi L}+\frac{\phi C_3}{\psi}$

$\phi = 12EI/L^2GA$
$\psi = 1+\phi$

Consistent Mass Matrix

	U_1	W_1	Θ_1	U_2	W_2	Θ_2
U_1	$\frac{mL}{3}$	0	$\frac{RL}{3}$	$\frac{mL}{6}$	0	$\frac{RL}{6}$
W_1		$\frac{mL}{3}$	0	0	$\frac{mL}{6}$	0
Θ_1			$\frac{mL^3}{120}+\frac{IL}{3}$	$\frac{RL}{6}$	0	$-\frac{mL^3}{120}+\frac{IL}{6}$
U_2				$\frac{mL}{3}$	0	$\frac{RL}{3}$
W_2		SYMMETRIC			$\frac{mL}{3}$	0
Θ_2						$\frac{mL^3}{120}+\frac{IL}{3}$

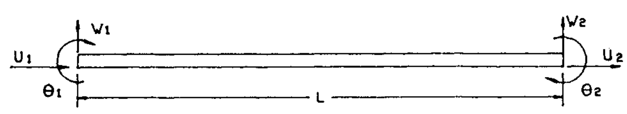

Fig. A1 Finite element for the extended Timoshenko beam model.

$$= \frac{1}{2}\{\dot{D}\}^T[M]_{ETB}\{\dot{D}\} \tag{A4b}$$

where $\{D\} = \{U_1, W_1, \Theta_1, U_2, W_2, \Theta_2\}$ is the continuum displacement vector and

$$m = \int_{-h/2}^{h/2} \rho \, dz, \; R = \int_{-h/2}^{h/2} \rho z \, dz, \; I = \int_{-h/2}^{h/2} \rho z^2 \, dz \tag{A5}$$

Substitution of the displacement fields, Eqs. (A3), into Eqs. (A4) yields the finite-element stiffness matrix $[K]_{ETB}$ and mass matrix $[M]_{ETB}$ of the model. When coupling terms C_i are relatively smaller than EA, GA, and EI, the element matrices can be approximated as in Table 5. Also note that they become identical to Przemienieck's results[27] when the elastic coupling terms vanish. From the finite-element matrices of Table 5, the reduced finite element matrices, $[\bar{K}]_{ETB}$ and $[\bar{M}]_{ETB}$, are derived by the use of reduction rules defined in Eqs. (12) and (13).

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